

INEQUALITIES FOR \log – CONVEX FUNCTIONS VIA THREE TIMES DIFFERENTIABILITY

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ABSTRACT. In this paper, we obtain some new integral inequalities like Hermite-Hadamard type for third derivatives absolute value are \log – convex. We give some applications to quadrature formula for midpoint error estimate.

1. INTRODUCTION

We shall recall the definitions of convex functions and \log – convex functions:

Let I be an interval in \mathbb{R} . Then $f : I \rightarrow \mathbb{R}$ is said to be convex if for all on $x, y \in I$ and all $\alpha \in [0, 1]$,

$$(1.1) \quad f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

holds. If (1.1) is strict for all $x \neq y$ and $\alpha \in (0, 1)$, then f is said to be strictly convex. If the inequality in (1.1) is reversed, then f is said to be concave. If it is strict for all $x \neq y$ and $\alpha \in (0, 1)$, then f is said to be strictly concave.

A function is called \log – convex or multiplicatively convex on a real interval $I = [a, b]$, if $\log f$ is convex, or, equivalently if for all $x, y \in I$ and all $\alpha \in [0, 1]$,

$$(1.2) \quad f(\alpha x + (1 - \alpha)y) \leq f(x)^\alpha + f(y)^{(1-\alpha)}.$$

It is said to be \log – concave if the inequality in (1.2) is reversed.

For some results for \log – convex functions see [1]-[4].

The following inequality is called Hermite-Hadamard inequality for convex functions:

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

holds.

The main purpose of this paper is to obtain some new integral inequalities like Hermite-Hadamard type for third derivatives absolute value are \log – convex.

In order to prove our main results for \log – convex functions we need the following Lemma from [5]:

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Lemma 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a three times differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(3)} \in L_1([a, b])$, then

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^3}{96} \left[\int_0^1 t^3 f^{(3)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt - \int_0^1 t^3 f^{(3)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt \right]. \end{aligned}$$

2. INEQUALITIES FOR \log -CONVEX FUNCTIONS

We shall start the following result:

Theorem 1. Let $f : I \rightarrow [0, \infty)$, be a three times differentiable mapping on I° such that $f''' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f'''|$ is \log -convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \{ |f'''(b)| \mu_K + |f'''(a)| \mu_M \} \end{aligned}$$

where

$$\begin{aligned} \mu_K &= \frac{2K^{\frac{1}{2}} (\ln K - 6)}{(\ln K)^2} + \frac{48K^{\frac{1}{2}} (\ln K - 2)}{(\ln K)^4} + \frac{96}{(\ln K)^4}, \\ \mu_M &= \frac{2M^{\frac{1}{2}} (\ln M - 6)}{(\ln M)^2} + \frac{48M^{\frac{1}{2}} (\ln M - 2)}{(\ln M)^4} + \frac{96}{(\ln M)^4} \end{aligned}$$

and

$$K = \frac{|f'''(a)|}{|f'''(b)|}, \quad M = \frac{|f'''(b)|}{|f'''(a)|}.$$

Proof. From Lemma 1, property of the modulus and \log -convexity of $|f'''|$ we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t^3 \left| f''' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right| dt + \int_0^1 t^3 \left| f''' \left(\frac{t}{2}b + \frac{2-t}{2}a \right) \right| dt \right\} \\ & \leq \frac{(b-a)^3}{96} \left\{ \int_0^1 t^3 |f'''(a)|^{\frac{t}{2}} |f'''(b)|^{1-\frac{t}{2}} dt + \int_0^1 t^3 |f'''(b)|^{\frac{t}{2}} |f'''(a)|^{1-\frac{t}{2}} dt \right\} \\ & = \frac{(b-a)^3}{96} \left\{ |f'''(b)| \int_0^1 t^3 \left[\frac{|f'''(a)|}{|f'''(b)|} \right]^{\frac{t}{2}} dt + |f'''(a)| \int_0^1 t^3 \left[\frac{|f'''(b)|}{|f'''(a)|} \right]^{\frac{t}{2}} dt \right\}. \end{aligned}$$

The proof is completed by making use of the necessary computation. \square

Corollary 1. Let μ_K, μ_M, K and M be defined as in Theorem 1. If we choose $f''\left(\frac{a+b}{2}\right) = 0$ in Theorem 1, we obtain the following inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \{ |f'''(b)| \mu_K + |f'''(a)| \mu_M \}. \end{aligned}$$

Theorem 2. Let $f : I \rightarrow [0, \infty)$, be a three times differentiable mapping on I° such that $f''' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ is $\log -$ convex on $[a, b]$, then the following inequality holds for some fixed $q > 1$

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left(\frac{1}{3p+1} \right)^{\frac{1}{p}} \left\{ |f'''(b)| \left(\frac{2}{q \ln K} [K^{\frac{q}{2}} - 1] \right)^{\frac{1}{q}} + |f'''(a)| \left(\frac{2}{q \ln M} [M^{\frac{q}{2}} - 1] \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $K = \frac{|f'''(a)|}{|f'''(b)|}$, $M = \frac{|f'''(b)|}{|f'''(a)|}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and using the Hölder's integral inequality, we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left\{ \left(\int_0^1 t^{3p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f''' \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 t^{3p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f''' \left(\frac{t}{2} b + \frac{2-t}{2} a \right) \right|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

If we use the \log -convexity of $|f'''|^q$ above, we can write

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left\{ \left(\int_0^1 t^{3p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'''(a)|^{\frac{qt}{2}} |f'''(b)|^{q-\frac{qt}{2}} dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 t^{3p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'''(b)|^{\frac{qt}{2}} |f'''(a)|^{q-\frac{qt}{2}} dt \right)^{\frac{1}{q}} \right\} \\ & = \frac{(b-a)^3}{96} \left(\frac{1}{3p+1} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ |f'''(b)| \left(\frac{2}{q \ln K} [K^{\frac{q}{2}} - 1] \right)^{\frac{1}{q}} + |f'''(a)| \left(\frac{2}{q \ln M} [M^{\frac{q}{2}} - 1] \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where

$$\int_0^1 t^{3p} dt = \frac{1}{3p+1},$$

$$\int_0^1 |f'''(a)|^{\frac{qt}{2}} |f'''(b)|^{q-\frac{qt}{2}} dt = |f'''(b)|^q \left(\frac{2}{q \ln K} [K^{\frac{q}{2}} - 1] \right)$$

and

$$\int_0^1 |f'''(b)|^{\frac{qt}{2}} |f'''(a)|^{q-\frac{qt}{2}} dt = |f'''(a)|^q \left(\frac{2}{q \ln M} [M^{\frac{q}{2}} - 1] \right).$$

The proof is completed. \square

Corollary 2. *Let K and M be defined as in Theorem 2. If we choose $f''\left(\frac{a+b}{2}\right) = 0$ in Theorem 2, we obtain the following inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left(\frac{1}{3p+1} \right)^{\frac{1}{p}} \left\{ |f'''(b)| \left(\frac{2}{q \ln K} \left[K^{\frac{q}{2}} - 1 \right] \right)^{\frac{1}{q}} + |f'''(a)| \left(\frac{2}{q \ln M} \left[M^{\frac{q}{2}} - 1 \right] \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3. *Let $f : I \rightarrow [0, \infty)$, be a three times differentiable mapping on I° such that $f''' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f'''|^q$ is log-convex on $[a, b]$, then the following inequality holds for some fixed $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left(\frac{1}{4} \right)^{1-\frac{1}{q}} \left\{ |f'''(b)| (\mu_{K,q})^{\frac{1}{q}} + |f'''(a)| (\mu_{M,q})^{\frac{1}{q}} \right\} \\ & \mu_{K,q} = \frac{2K^{\frac{q}{2}} (q \ln K - 6)}{(q \ln K)^2} + \frac{48K^{\frac{q}{2}} (q \ln K - 2)}{(q \ln K)^4} + \frac{96}{(q \ln K)^4}, \\ & \mu_{M,q} = \frac{2M^{\frac{q}{2}} (q \ln M - 6)}{(q \ln M)^2} + \frac{48M^{\frac{q}{2}} (q \ln M - 2)}{(q \ln M)^4} + \frac{96}{(q \ln M)^4} \end{aligned}$$

and

$$K = \frac{|f'''(a)|}{|f'''(b)|}, \quad M = \frac{|f'''(b)|}{|f'''(a)|}.$$

Proof. From Lemma 1, using the well-known power-mean integral inequality and log-convexity of $|f'''|^q$ we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left\{ \left(\int_0^1 t^3 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^3 \left| f''' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 t^3 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^3 \left| f''' \left(\frac{t}{2}b + \frac{2-t}{2}a \right) \right|^q dt \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(b-a)^3}{96} \left\{ \left(\int_0^1 t^3 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^3 |f'''(a)|^{\frac{qt}{2}} |f'''(b)|^{q-\frac{qt}{2}} dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 t^3 dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^3 |f'''(b)|^{\frac{qt}{2}} |f'''(a)|^{q-\frac{qt}{2}} dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

The proof is completed by making use of the necessary computation. \square

Corollary 3. Let $\mu_{K,q}, \mu_{M,q}, K$ and M be defined as in Theorem 3. If we choose $f''\left(\frac{a+b}{2}\right) = 0$ in Theorem 3, we obtain the following inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)^3}{96} \left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left\{ |f'''(b)| (\mu_{K,q})^{\frac{1}{q}} + |f'''(a)| (\mu_{M,q})^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 4. From Corollaries 1-3, we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \min \{\chi_1, \chi_2, \chi_3\}$$

where

$$\begin{aligned} \chi_1 &= \frac{(b-a)^3}{96} \left\{ |f'''(b)| \frac{2K^{\frac{1}{2}} (\ln K - 6)}{(\ln K)^2} + \frac{48K^{\frac{1}{2}} (\ln K - 2)}{(\ln K)^4} + \frac{96}{(\ln K)^4} \right. \\ & \quad \left. + |f'''(a)| \frac{2M^{\frac{1}{2}} (\ln M - 6)}{(\ln M)^2} + \frac{48M^{\frac{1}{2}} (\ln M - 2)}{(\ln M)^4} + \frac{96}{(\ln M)^4} \right\}, \\ \chi_2 &= \frac{(b-a)^3}{96} \left(\frac{1}{3p+1}\right)^{\frac{1}{p}} \\ & \quad \times \left\{ |f'''(b)| \left(\frac{2}{q \ln K} [K^{\frac{q}{2}} - 1]\right)^{\frac{1}{q}} + |f'''(a)| \left(\frac{2}{q \ln M} [M^{\frac{q}{2}} - 1]\right)^{\frac{1}{q}} \right\}, \\ \chi_3 &= \frac{(b-a)^3}{96} \left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left\{ |f'''(b)| \left(\frac{2K^{\frac{q}{2}} (q \ln K - 6)}{(q \ln K)^2} + \frac{48K^{\frac{q}{2}} (q \ln K - 2)}{(q \ln K)^4} + \frac{96}{(q \ln K)^4}\right)^{\frac{1}{q}} \right. \\ & \quad \left. + |f'''(a)| \left(\frac{2M^{\frac{q}{2}} (\ln M - 6)}{(\ln M)^2} + \frac{48M^{\frac{q}{2}} (\ln M - 2)}{(\ln M)^4} + \frac{96}{(\ln M)^4}\right)^{\frac{1}{q}} \right\} \end{aligned}$$

and $K = \frac{|f'''(a)|}{|f'''(b)|}$, $M = \frac{|f'''(b)|}{|f'''(a)|}$.

Remark 1. In Theorem 3 and Corollary 3, if we choose $q = 1$, we obtain Theorem 1 and Corollary 1 respectively.

3. APPLICATIONS TO MIDPOINT FORMULA

We give some error estimates to midpoint formula by using the results of Section 2.

Let d be a division $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the formula

$$\int_a^b f(x) dx = M(f, d) + E(f, d)$$

where $M(f, d) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) (x_{i+1} - x_i)$ for the midpoint version and $E(f, d)$ denotes the associated approximation error.

Proposition 1. Let $f : I \rightarrow [0, \infty)$ be a three times differentiable mapping on I° with $a, b \in I^\circ$ such that $a < b$. If $|f'''|$ is log-convex function with $f''' \in L_1([a, b])$, then for every division d of $[a, b]$, the midpoint error estimate satisfies

$$|E(f, d)| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^4}{96} \{ |f'''(x_{i+1})| \mu_1 + |f'''(x_i)| \mu_2 \}$$

where

$$\begin{aligned} \mu_1 &= \frac{2K_1^{\frac{1}{2}} (\ln K_1 - 6)}{(\ln K_1)^2} + \frac{48K_1^{\frac{1}{2}} (\ln K_1 - 2)}{(\ln K_1)^4} + \frac{96}{(\ln K_1)^4}, \\ \mu_2 &= \frac{2M_1^{\frac{1}{2}} (\ln M_1 - 6)}{(\ln M_1)^2} + \frac{48M_1^{\frac{1}{2}} (\ln M_1 - 2)}{(\ln M_1)^4} + \frac{96}{(\ln M_1)^4} \end{aligned}$$

and

$$K_1 = \frac{|f'''(x_i)|}{|f'''(x_{i+1})|}, \quad M_1 = \frac{|f'''(x_{i+1})|}{|f'''(x_i)|}.$$

Proof. By applying Corollary 1 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, \dots, n-1)$ of the division d , we have

$$\begin{aligned} & \left| \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) dx - f\left(\frac{x_i + x_{i+1}}{2}\right) \right| \\ & \leq \frac{(x_{i+1} - x_i)^3}{96} \{ |f'''(x_{i+1})| \mu_1 + |f'''(x_i)| \mu_2 \}. \end{aligned}$$

By summing over i from 0 to $n-1$, we can write

$$\left| \int_a^b f(x) dx - M(f, d) \right| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^4}{96} \{ |f'''(x_{i+1})| \mu_1 + |f'''(x_i)| \mu_2 \}$$

which completes the proof. \square

Proposition 2. Let $f : I \rightarrow [0, \infty)$ be a three times differentiable mapping on I° with $a, b \in I^\circ$ such that $a < b$. If $|f'''|^q$ is log-convex function with $f''' \in L_1([a, b])$ for some fixed $q > 1$, then for every division d of $[a, b]$, the midpoint error estimate satisfies

$$\begin{aligned} |E(f, d)| & \leq \left(\frac{1}{3p+1} \right)^{\frac{1}{p}} \frac{1}{96} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left\{ |f'''(x_{i+1})| \left(\frac{2}{q \ln K_1} [K_1^{\frac{q}{2}} - 1] \right)^{\frac{1}{q}} \right. \\ & \quad \left. + |f'''(x_i)| \left(\frac{2}{q \ln M_1} [M_1^{\frac{q}{2}} - 1] \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and K_1, M_1 are as defined in Proposition 1.

Proof. The proof can be maintained by using Corollary 2 like Proposition 1. \square

Proposition 3. Let $f : I \rightarrow [0, \infty)$ be a three times differentiable mapping on I° with $a, b \in I^\circ$ such that $a < b$. If $|f'''|^q$ is log-convex function with $f''' \in L_1([a, b])$ for some fixed $q \geq 1$, then for every division d of $[a, b]$, the midpoint error estimate satisfies

$$|E(f, d)| \leq \frac{1}{96} \left(\frac{1}{4} \right)^{1-\frac{1}{q}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left\{ |f'''(x_{i+1})| (\mu_{1,q})^{\frac{1}{q}} + |f'''(x_i)| (\mu_{2,q})^{\frac{1}{q}} \right\}$$

where

$$\mu_{1,q} = \frac{2K_1^{\frac{q}{2}}(q \ln K_1 - 6)}{(q \ln K_1)^2} + \frac{48K_1^{\frac{q}{2}}(q \ln K_1 - 2)}{(q \ln K_1)^4} + \frac{96}{(q \ln K_1)^4},$$

$$\mu_{2,q} = \frac{2M_1^{\frac{q}{2}}(q \ln M_1 - 6)}{(q \ln M_1)^2} + \frac{48M_1^{\frac{q}{2}}(q \ln M_1 - 2)}{(q \ln M_1)^4} + \frac{96}{(q \ln M_1)^4}$$

and K_1, M_1 are as defined in Proposition 1.

Proof. The proof can be maintained by using Corollary ?? like Proposition 1. \square

REFERENCES

- [1] M. Alomari and M. Darus, On the Hadamard's inequality for $\log -$ convex functions on the coordinates, Journal of Inequalities and Applications, Volume 2009, Article ID 283147, 13 pages.
- [2] X. Zhang and W. Jiang, Some properties of $\log -$ convex function and applications for the exponential function, Computers and Mathematics with Applications 63 (2012) 1111–1116.
- [3] B. G. Pachpatte, A note on integral inequalities involving two \log -convex functions, Mathematical Inequalities & Applications, vol. 7, no. 4, pp. 511–515, 2004.
- [4] J. Pečarić and A. U. Rehman, On logarithmic convexity for power sums and related results, Journal of Inequalities and Applications, vol. 2008, Article ID 389410, 9 pages, 2008.
- [5] Y. Shuang, Y. Wang and F. Qi, Some inequalities of Hermite-Hadamard type for functions whose third derivatives are (α, m) -convex, J. Computational Analysis and Applications, Vol. 17, No:2, 2014.

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